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1998 J. Phys. A: Math. Gen. 31 4521

(http://iopscience.iop.org/0305-4470/31/19/015)

J. Phys. A: Math. Gen. **31** (1998) 4521–4531. Printed in the UK PII: S0305-4470(98)89163-4

# Non-commutative polynomials and the transport properties in multichannel-multilayer systems

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Received 11 November 1997, in final form 6 February 1998

**Abstract.** Using the transfer-matrix technique to describe transport properties in multichannel-multilayer systems, a three-term non-commutative matrix recurrence relation is deduced and solved. The matrix polynomials obtained in this way allow one to write compact expressions for the $n$-cell transmission amplitudes, $(t_n)_{ij}$, from channel $j$ to channel $i$. In the one-dimensional, one-channel limit, the non-commutative polynomials reduce to the well known Chebyshev orthogonal polynomials. To illustrate the role of these polynomials in the resonant tunnelling and channel-mixing behaviour, we discuss one- and two-channel examples.

## 1. Introduction

Recent developments in thin-film epitaxial growth techniques have generated increasing interest in theoretical and experimental research on transport properties of finite multilayer systems. These properties have been investigated in a variety of arrays of alternating thin layers with different bandgaps. However, most of the studied heterostructures contain just a few layers, i.e. double- and triple-barrier systems [1–4].

Well established theories like Bloch's theorem, diagrammatic Green function techniques and some general properties of macroscopic ordered systems have been useful in the understanding of transport properties of metals and semiconductors. At the mesoscopic scale of locally periodic heterostructures, these methods, applied to evaluate transmission coefficients in quasi one-dimensional (1D) systems [5], were restricted to rather few propagating modes, mainly one. As the number of propagating modes grows or the number of layers increases, any analytical description within these approaches becomes impossible. It is for this kind of system that the method discussed here could be appropriate. Some properties appearing when more than one propagating mode is present, like the resonant tunnelling, band structure and channel-mixing effects, seem to be better described in terms of orthogonal non-commutative polynomials.

In nanostructured devices with finite transverse section more than one physical state is normally involved. In electronic scattering theory [6] the transverse quantum states have been useful to introduce multichannel transfer matrices. Normally, channels are defined depending on the physical model envisioned. In this sense, light and heavy holes, or any other propagating modes, are physical realizations of channels.

Using well known transfer-matrix properties together with the local periodicity of these systems, simple recurrence relations for the transfer-matrix blocks are obtained. From

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these relations, both compact formulae for the relevant scattering amplitudes and a non-commutative three-term recurrence relation are deduced. The recurrence relation is solved using the generating function method and a new set of non-commutative polynomials is obtained. In the 1D limit, with only one propagating mode, the non-commutative polynomials reduce to the well known Chebyshev orthogonal polynomials of the second class. From the physical point of view, it is important to note that all of the information of the complicated, although coherent, scattering process due to multiple reflections and interfering phenomena along the *n*-cells of the superlattice, is contained in these polynomials. This is particularly clear in the 1D one-channel case where the band structure and resonant behaviour is strongly determined by the Chebyshev polynomial properties. This is shown in the first example of section 4.

To define the kind of systems which we are interested in, we outline in section 2 some well known quantum results and establish new scattering relations, which we claim are useful and appropriate to describe tunnelling properties in finite periodic systems. In section 3, new and compact formulae for the $N$-channel (two and four probe) Landauer conductance, and a new set of non-commutative orthogonal polynomials are deduced. In the last section, we give some simple examples. In the one-channel limit, we deal first with a very general example. Without specifying any particular 1D potential profile, we write the transfer matrix in a form which is common to each and every one of the specific potential profiles, i.e. we take the transfer matrix in the Bargmann representation and analyse the transmission resonances and the band structure in terms of Bargmann parameters. In this general representation, it is absolutely clear that in all of the 1D one-channel superlattices, the resonant behaviour bears a strong relation to the associated Chebyshev polynomial behaviour. We also apply our results to specific and familiar potential shapes: the square- and the $\delta$-barrier potentials. To illustrate the use of this method in a multimode transport process with channel mixing, we consider a simple two-channel bilayer sequence $ABABA\dots B$, with $B$ a monolayer of $\delta$-scatterers. Though the results presented here are appropriate to evaluate transport properties through multilayer-multichannel, time-reversal-invariant superlattices, the method can also be applied to other universality classes. Our results extend easily to systems under a magnetic field.

## 2. The transfer matrix and the matrix recurrence relation

In the scattering approach to transport processes, the transfer matrix $M$ connecting wavevectors at the left- and right-hand sides of the scatterer system contains all the information required to calculate reflection and transmission amplitudes. The same applies to the transfer matrix $M_{fd}$ which relates wavefunctions and their derivatives, mostly used in solid-state physics and related to $M$ by a simple unitary transformation.

In order to define what we mean by a multichannel transfer matrix, let us consider an electronic transport process through a 3D system (of length $l = z_{\rm R} - z_{\rm L}$, and transverse cross section $w_x w_y$), connected to perfect leads (or waveguides) of equal cross section. Due to the finiteness of this cross section, the potential function can be separated into, at least, two parts: a transverse hard wall potential $V_{\rm T}(x, y)$, independent of the growing coordinate $z$, and a potential $V_{\rm P}(x, y, z)$, periodic at least as a function of $z$. For a given Fermi energy $E_{\rm F}$, the transverse eigenfunctions $\phi_{n_x n_y}(x, y)$ and the eigenvalues $E_{n_x n_y}$ are easily obtained in the leads and those regions where $V_{\rm P} = 0$. The electrons with energy $E = E_{\rm F}$ populating these transverse modes, propagate along the direction $z$ as plane waves with longitudinal wavenumbers

$$k_{z,i}^2 = \frac{2m}{\hbar^2}E - \pi^2\left(\frac{n_x^2}{w_x^2} + \frac{n_y^2}{w_y^2}\right) = k^2 - k_{\mathrm{T},i}^2.$$

Each pair $\{n_x, n_y\}$ defines a channel, considered non-evanescent when $k_{z,i}$ is real. If we write the Schrödinger wavefunction as

$$\psi(x, y, z) = \sum_i^N \varphi_i(z)\phi_i(x, y) \qquad \text{for } z_\mathrm{L} \leqslant z \leqslant z_\mathrm{R}$$

where $N$ is the number of propagating modes, we obtain the well known system of coupled equations [7]

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\varphi_i(z) + (k^2 - k_{\mathrm{T}i}^2)\varphi_i(z) = \sum_{j=1}^N K_{ij}\varphi_j(z) \tag{1}$$

with channel coupling constants defined by

$$K_{ij} = \frac{2m_\mathrm{e}^*}{w_x w_y \hbar^2}\int_0^{w_y}\int_0^{w_x} \phi_i^*(x, y)V_\mathrm{P}(x, y, z)\phi_j(x, y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2}$$

In this way, the 3D multichannel problem is reduced into a 1D multichannel problem. Using the transfer matrix method, we will go further and reduce the problem into a one-cell 1D multichannel problem.

Let us consider a single cell extending from $z_1$ to $z_2$. Assume also that at these points the propagating wavefunctions can be written as

$$\psi(z_1) = \sum_i [a_i \overrightarrow{\varphi}_i(z_1) + b_i \overleftarrow{\varphi}_i(z_1)]$$

and

$$\psi(z_2) = \sum_i [c_i \overrightarrow{\varphi}_i(z_2) + d_i \overleftarrow{\varphi}_i(z_2)]$$

with $\overrightarrow{\varphi_i}$ and $\overleftarrow{\varphi_i}$ travelling waves (in channel $i$) to the right and left, respectively. To define the transfer matrix, which connects wavevectors between $z_1$ and $z_2$, it is convenient to write the previous wavefunctions as $2N$-dimensional vectors. So, we have

$$\phi_\mathrm{L}(z_1) = \begin{pmatrix} \overrightarrow{\phi}_\mathrm{L}(z_1) \\ \overleftarrow{\phi}_\mathrm{L}(z_1) \end{pmatrix} \qquad \text{with } \overrightarrow{\phi}_\mathrm{L}(z) = \begin{pmatrix} a_1\overrightarrow{\varphi}_1(z) \\ \vdots \\ a_N\overrightarrow{\varphi}_N(z) \end{pmatrix}$$

$$\text{and } \overleftarrow{\phi}_\mathrm{L}(z) = \begin{pmatrix} b_1\overleftarrow{\varphi}_1(z) \\ \vdots \\ b_N\overleftarrow{\varphi}_N(z) \end{pmatrix} \tag{3}$$

and similar ones at $z_2$ and inside the cell. Finiteness and smooth matching requirements lead to

$$\phi_\mathrm{R}(z_2) = M(z_1, z_2)\phi_\mathrm{L}(z_1) \tag{4}$$

where $M(z_1, z_2)$, or just $M$, is the single-cell $N$-channel transfer matrix of dimension $2N \times 2N$ and complex entries. Although transfer matrices are clearly defined, the explicit calculation for more than one channel is not generally straightforward. It may require powerful mathematical techniques, see for example [8].

In general, for a time reversal invariant (TRI) and *spin-independent* scattering processes (i.e. for systems of the orthogonal universality class) the transfer matrices have the following structure

$$M_{\rm o} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \tag{5}$$

with $\alpha$ and $\beta$, $N \times N$ complex sub-matrices. For TRI and *spin-dependent* scattering processes (of the symplectic universality class), the matrices take the form [9]

$$M_s = \begin{pmatrix} \alpha_s & \beta_s \\ \kappa\beta_s^*\kappa^{\rm T} & \kappa\alpha_s^*\kappa^{\rm T} \end{pmatrix} \qquad \text{with} \qquad \kappa = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

where $\alpha_s$ and $\beta_s$, are $2N \times 2N$ complex sub-matrices for spin-$\frac{1}{2}$ particles. In the following, we shall refer only to the orthogonal class, thus the subindex o will not be used in the $M$ matrix.

Additionally, it is known that flux conservation (FC) requires the fulfillment of

$$M\Sigma_z M^\dagger = \Sigma_z \tag{6}$$

where $\Sigma_z$ is the generalized $\sigma_z$ Pauli matrix. This requirement has, of course, consequences on the final symmetries and possible representations of the transfer matrix [9]. A particular representation compatible with FC, is the Bargmann's parametrization

$$M = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}$$

where $u$ and $v$ are unitary matrices, and $\chi$ a positive diagonal matrix.

Since we are interested in obtaining transmission amplitudes, we recall that the transfer matrix, in the orthogonal universality class, can also be written as [6]

$$M = \begin{pmatrix} (t^\dagger)^{-1} & -(t^\dagger)^{-1}r^* \\ -(t^{\rm T})^{-1}r & (t^{\rm T})^{-1} \end{pmatrix} \tag{7}$$

with $t$ and $r$ the transmission and reflection amplitudes.

If we put two identical layers of length $L/n$ and transfer matrices $M$ side by side, the resulting system has length $2L/n$, and transfer matrix $M_2 = MM$. The repeated application of this combination property leads us to express the transfer matrix, for the $n$ identical-cell system, as

$$M_n = M^n = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}^n \equiv \begin{pmatrix} \alpha_n & \beta_n \\ \beta_n^* & \alpha_n^* \end{pmatrix}. \tag{8}$$

Our main objective is to express $\alpha_n$ and $\beta_n$ as simple functions of $\alpha$ and $\beta$, and then to obtain simple functions for the multilayer transmission amplitudes. For this purpose we shall deduce and solve some recurrence relations. From $M_n = MM_{n-1}$, we have

$$\begin{aligned} \alpha_n &= \alpha\alpha_{n-1} + \beta\beta_{n-1}^* \\ \beta_n &= \alpha\beta_{n-1} + \beta\alpha_{n-1}^* \end{aligned} \tag{9}$$

and obtain

$$\begin{aligned} &\alpha_{n+1} - A\alpha_n + B\alpha_{n-1} = 0 \qquad \text{with } \alpha_0 = I_N \\ &\beta_{n+1} - A\beta_n + B\beta_{n-1} = 0 \qquad \text{with } \beta_0 = 0. \end{aligned} \tag{10}$$

Here, $A = \alpha + \beta\alpha^*\beta^{-1}$ and $B = \beta\left(\alpha^*\beta^{-1}\alpha - \beta^*\right)$. We find it convenient to define the function

$$p_{N,n} = \beta^{-1}\beta_{n+1} \tag{11}$$

which for simplicity will be written just as $p_n$, unless the number of channels $N$ needs to be specified. Using these functions we get the very important matrix recurrence relation (MRR)

$$p_n - \zeta p_{n-1} + \eta p_{n-2} = 0 \qquad \text{for } n \geqslant 1 \tag{12}$$

where $p_{-1} = 0$, $p_0 = I_N$ (the unit matrix of dimension $N$), $\zeta = (\beta^{-1}\alpha\beta + \alpha^*)$ and $\eta = (\alpha^*\beta^{-1}\alpha\beta - \beta^*\beta)$. It is easy to see that for the one-channel case the MRR is just the recurrence relation for Chebyshev polynomials. In the next section, this matrix recurrence relation will be solved and the physical quantities will be expressed in terms of its solution: the non-commutative polynomials $p_n$.

## 3. The scattering amplitudes and the non-commutative polynomials

The non-commutative relation (12) resembles the well known three-term orthogonal-polynomial recurrence relations. The non-commutative character of the various factors involved makes the problem not only interesting from the physical but also from the mathematical point of view, as there is a lot of ongoing research on non-commutative symmetric functions [10]. As mentioned before, in the particular case of $N = 1$ we have the recurrence relation for the Chebyshev polynomials of the second kind, $U_n((\alpha + \alpha^*)/2)$. In this case, we find (see [11]) previous work consistent with our results. In terms of the polynomials $p_n$, the $n$-cell transmission amplitude, $t_n$, and the four-probe Landauer conductance both take compact forms. Using $\beta_n = \beta p_{n-1}$, and $t_n = (\alpha_n^\dagger)^{-1}$, we immediately obtain

$$t_n^{\mathrm{T}} = (p_n - (\beta^{-1}\alpha\beta)p_{n-1})^{-1}. \tag{13}$$

Having $t$, the two-probe Landauer conductance, $\mathcal{G}_n = \mathrm{Tr}(tt^\dagger)$ in units of $e^2/\pi\hbar$, is obvious. Moreover, for the four-probe Landauer conductance, $G_n$, we have

$$G_n = \frac{1}{p_{n-1}} G \left(\frac{1}{p_{n-1}}\right)^\dagger. \tag{14}$$

Here $G$ is the single-cell conductance $|t/r|^2$, in units of $e^2/\pi\hbar$. Both $t_n$ (thus $\mathcal{G}_n$) and $G_n$ are *simple functions of the polynomial* $p_{n-1}$, and the single-cell physical quantities. It is now clear, as mentioned before, that the polynomial's zeros determine the transmission and conductance resonances. The information on the multiple reflection from the conduction band discontinuities, and phase interference phenomena along the system, is also carried by these polynomials. Our next aim is to solve the MRR and to obtain the $p_n$ polynomials.

Although the MRR can be directly solved, we found it convenient to transform it into a recurrence relation with scalar coefficients (though many more terms). As could be expected, the new recurrence relation takes a form completely consistent with the Cayley–Hamilton theorem applied to $M$. In this case we have

$$M_{n+2N} + g_1 M_{n+2N-1} + \cdots + g_{2N-1} M_{n+1} + g_{2N} M_n = 0 \tag{15}$$

where $g_m$ are scalar symmetric homogeneous functions. This *scalar recurrence relation* (SRR) can, of course, be written with $\beta_m$ or $\alpha_m$ instead of $M_m$, and also for the polynomials $p_m$ introduced before as

$$p_{n+2N} + g_1 p_{n+2N-1} + \cdots + g_{2N} p_n = 0 \tag{16}$$

with $n \geqslant 0$, and the same coefficients $g_m$.

We shall first consider the generating function

$$Q(\lambda) = \frac{I_N}{1 + g_1\lambda + g_2\lambda^2 + \cdots + g_{2N}\lambda^{2N}} = \sum_{m=0} q_m \lambda^m \tag{17}$$

where $\lambda$ is a complex matrix (for example, the diagonal matrix of the eigenvalues $\lambda_i$ of $M$). The coefficients $q_m$ are just complex scalars. They satisfy the SRR and the $2N$ conditions

$$\sum_{j=0}^{k} g_j q_{k-j} = \delta_{k,0} \qquad k = 0, 1, \ldots, 2N-1 \tag{18}$$

but they do not satisfy the MRR, except in the one-channel case. Using (12) we can obtain $p_0$, $p_1$, ..., and $p_{2N-1}$. We call these functions the initial conditions. They are different from $q_0$, $q_1$, ..., and $q_{2N-1}$, obtained from (18). Even with this observation, the $q_m$ are still useful. Since the $\lambda_i$ are solutions of the scalar recurrence relation, we can take the combination

$$q_n = s_1\lambda_1^n + s_2\lambda_2^n + \cdots + s_{2N}\lambda_{2N}^n \tag{19}$$

and determine the coefficients $s_i$. If we replace this combination in (18), we have

$$\sum_{i=1}^{2N} d_{ki} s_i = \delta_{k,0} \qquad k = 0, 1, \ldots, 2N-1 \tag{20}$$

with

$$d_{ki} = \lambda_i^k + g_1\lambda_i^{k-1} + \cdots + g_{k-1}\lambda_i + g_k. \tag{21}$$

Now we can use the well known symmetric homogeneous functions

$$g_m = \sum_{l_1<l_2<\cdots<l_m}^{2N} \lambda_{l_1}\lambda_{l_2}\ldots\lambda_{l_m} \tag{22}$$

and obtain

$$s_i = \frac{\lambda_i^{2N-1}}{\prod_{j\neq i}^{2N}(\lambda_i - \lambda_j)}. \tag{23}$$

Thus,

$$q_n = \sum \frac{\lambda_i^{2N+n-1}}{\prod_{j\neq i}^{2N}(\lambda_i - \lambda_j)}. \tag{24}$$

This is not yet the polynomial we are looking for; however, for $N = 1$ this scalar polynomial reduces to the Chebyshev polynomial $U_n$ evaluated at $\mathrm{Tr}\, M/2$. To fulfil the proper initial conditions (defined by the MRR) one has to consider the generating function

$$(I_N + \rho_1\lambda + \rho_2\lambda^2 + \cdots + \rho_{2N-1}\lambda^{2N-1})\; Q(\lambda) = \sum_{m=0} p_m\lambda^m \tag{25}$$

with $p_m = \sum_{k=0}^{m} \rho_k q_{m-k}$, when $m \leqslant 2N-1$, and $p_{m=} \sum_{k=0}^{2N-1} \rho_k q_{m-k}$, when $m \geqslant 2N$.

The MRR initial conditions are satisfied if and only if

$$\rho_k = p_k + g_1 p_{k-1} + \cdots + g_k p_0 \qquad k = 1, \ldots, 2N-1 \tag{26}$$

therefore

$$p_{N,m} = \sum_{k=0}^{2N-1} \sum_{l=0}^{k} p_l g_{k-l} q_{m-k} \qquad \text{when } m \geqslant 2N. \tag{27}$$

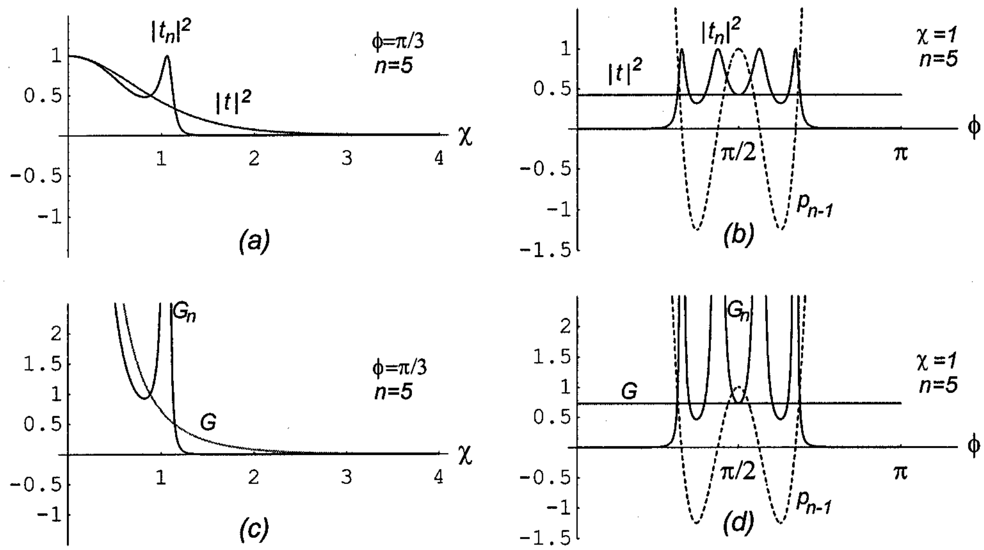


**Figure 1.** One- and $n$-cell Landauer conductances and transmission probabilities as functions of the free parameters $\phi = \phi_u - \phi_v$ and $\chi$, for arbitrary one-channel potential functions. In ($a$) and ($c$) $\phi$ is kept fixed, while in ($b$) and ($d$) $\chi$ is kept fixed. For specific potential profiles, these parameters are energy dependent. In ($b$) and ($d$), the polynomial $p_{1,4}$ is also shown. Clearly, its zeros determine the resonances.

This is one of our main results. It gives the non-commutative polynomials $p_{N,m}$ in terms of the invariant functions $g_m$ and $q_m$, and the first $2N-1$ polynomials $p_l$ obtained from the MRR. Once having the polynomials $p_{N,n}$, tunnelling probabilities can be calculated including transverse channel mixing. The common polynomial factors suggest certain universality in the evolution through a superlattice. It will be of interest to extend this method to include electric fields.

## 4. Some physical examples

Let us now discuss one- and two-channel examples. In one-channel, 1D systems the transmission amplitude through the $n$-cell chain is obtained from

$$t_n = t^* \left( \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} t^* - \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \right)^{-1}. \tag{28}$$

Independently of the particular and specific potential profile, the single-cell transmission amplitude and the transfer-matrix eigenvalues can be written in the Bargmann representation as

$$t = \mathrm{e}^{\mathrm{i}\phi} \frac{1}{\cosh \chi} \tag{29}$$

and

$$\lambda_{1,2} = \cos\phi \cosh\chi \pm \sqrt{(\cos\phi\cosh\chi)^2 - 1} \tag{30}$$

with $\phi \in [0, 2\pi)$ and $\chi \in [0, \infty)$ being the Bargmann parameters. Let us look at the behaviour of $|t_n|^2$ and $G_n$ as functions of these parameters. In figure 1($a$), the transmission probabilities $|t|^2$ and $|t_n|^2$ are shown as functions of $\chi$, when $\phi = 1$, whereas in figure 1($b$) the same quantities are plotted as functions of $\phi$ but now keeping $\chi = 1$. In figures 1($c$) and 1($d$) the Landauer conductances $G$ and $G_n$ are plotted for the same conditions.

To exhibit the role of the Chebyshev polynomial in the resonant behaviour of $|t_n|^2$ and $G_n$, we also plot the former in figures 1($b$) and 1($d$). The polynomials $p_{1,n-1}$ clearly determine the position and width of the allowed bands and tunnelling resonances. This is an interesting result, particularly when dealing with $N$-channel superlattices, where the matrix polynomials $p_{N,n}$ are closely linked to the channel-mixing probabilities $|t_{N,ij}|^2$.

In figure 2, we plot transmission probabilities for 1D square- and $\delta$-barrier potential chains, as functions of the energy. More precisely, we calculate the transmission probabilities between the left-hand side of the first, and the right-hand side of the last layer $B$ in the sequence $A'BABA\ldots ABA'$. Here $B$ is either a square or a delta barrier, and the $A'$ are layers of the same material as $A$, but different thickness. In this figure, we have taken $n = 11$ (i.e. 11 barriers). In figure 2($a$) the transmission probabilities $|t|^2$, $|t_n|^2$, and the bandwidth prediction for square barriers of height $V_s = 0.23$ eV and width $a_s = 30$ Å, separated by a distance $b_s = 80$ Å, are shown. In figure 2($b$) the same quantities are plotted, but now for $\delta$-barrier potentials, of strength $V_\delta = 0.13$ eV separated by a distance $b_\delta = 130$ Å from each other.

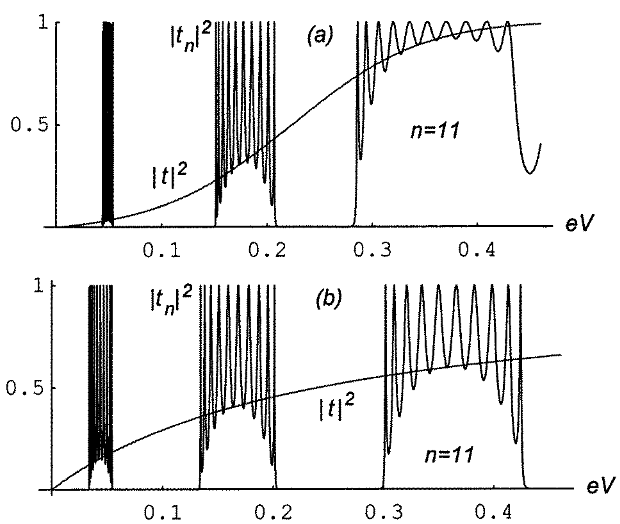


**Figure 2.** One-channel transmission probabilities $|t|^2$ and $|t_n|^2$ for $n = 11$ square- and $\delta$-barrier potentials, as functions of the energy. In ($a$) the square barriers are of height $V_s = 0.23$ eV and width $a_s = 30$ Å, and the valleys of width $b_s = 80$ Å. In ($b$) the strength of the $\delta$-barriers, separated by a distance $b_\delta = 130$ Å, is $V_\delta = 0.13$ eV.

If we have more than one open channel, and we assume no incidence in the extreme right-hand side, $t_{n,ik}$ represents the transmission amplitude from channel $k$ on the left to channel $i$ in the right. Hence $T_{n,i} = \sum_k |t_{n,ik}|^2$ is the total transmission probability to channel $i$.

To illustrate the evaluation of tunnelling properties for more than one channel, we shall consider the potential function $V = V_L + V_T$, where the transverse part $V_T$ is an infinite

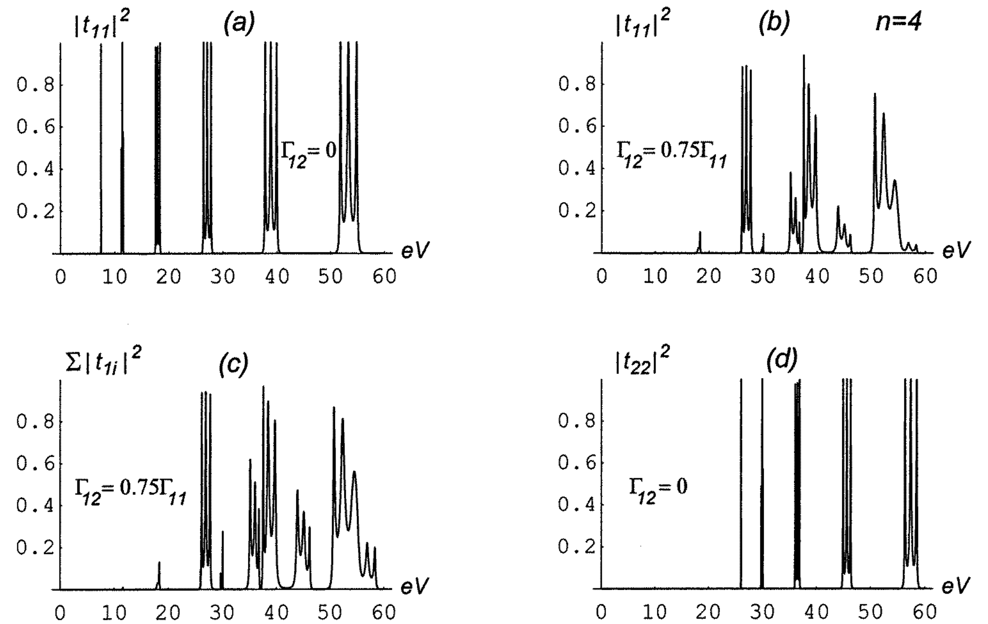


**Figure 3.** Some coupled and uncoupled two-channel transmission probabilities through a chain of $\delta$-barriers, as functions of the energy. In the uncoupled limit, $|t_{n,11}|^2$ and $|t_{n,22}|^2$ behave as in the one-channel case. Here the transverse width is $\approx$30 Å, the separation $\approx$30 Å and $n = 4$. The one-channel threshold is $\approx$6.2 eV and the two-channel threshold is $\approx$24.7 eV. When $\Gamma_{12} \neq 0$, the band mixing is shown for $|t_{n,11}|^2$ and $|T_{n,1}|^2$ in ($b$) and ($c$), respectively.

square well of widths $w = w_x = w_y$, while the longitudinal part is given by

$$V_{\rm L} = \gamma\left[\delta(z - \nu l_c)\sum_{\mu}\delta(x - x_\mu)\delta(y - y_\mu)\right] \qquad \text{with } \nu = 1, \ldots, n.$$

In order to have $N$ open channels, we take energies having positive longitudinal wavenumbers $k_i^2 = k^2 - k_{{\rm T}i}^2,\ i = 1, \ldots, N$. Defining the coupling constants

$$\Gamma_{ij} = \frac{2m^*\gamma}{\hbar^2}\sum_{\mu}\phi_i^*(x_\mu, y_\mu)\phi_j(x_\mu, y_\mu) \tag{31}$$

the two-channel $\delta$-barrier transfer matrix is given by

$$M_\delta = \begin{pmatrix} \alpha_\delta & \beta_\delta \\ \beta_\delta^* & \alpha_\delta^* \end{pmatrix}$$

where

$$\alpha_\delta = I_2 + \beta_\delta \qquad \text{and} \qquad \beta_\delta = \frac{1}{2{\rm i}}\begin{pmatrix} \Gamma_{11}/k_1 & \Gamma_{12}/k_1 \\ \Gamma_{21}/k_2 & \Gamma_{22}/k_1 \end{pmatrix}. \tag{32}$$

Here $\Gamma_{12}k_2 = \Gamma_{21}k_1$, when flux is conserved. Selected calculations are shown in figure 3. In figures 3($a$) and 3($d$), the transmission probabilities $|t_{n,11}|^2 \equiv |t_{11}|^2$ and $|t_{n,22}|^2 \equiv |t_{22}|^2$ are plotted for $n = 4$ and $\Gamma_{12} = 0$. In this *uncoupled* limit, we have well defined resonant bands. Their particular positions and widths depend, of course, on the specific choice for the underlying parameters. As soon as the coupling is turned on, the channel mixing takes place. In figure 3($b$) the tunnelling probability $|t_{11}|^2$ is calculated again. This time, tunnelling resonances appear at energies which, in the uncoupled limit, were forbidden for channel 1 and allowed for channel 2. In figure 3(*c*) the total transmission $T_{4,1} = \sum_i |t_{1i}|^2$ to channel 1 is plotted.

## 5. Conclusions

Within the scattering approach and the transfer-matrix technique to describe $N$ propagating modes through locally periodic 3D systems, a three-term recurrence relation for non-commutative polynomials, as well as compact and closed formulae for transmission and conductance probabilities, have been deduced. We have solved the matrix recurrence relation using the generating function method, and a new set of non-commutative polynomials have been obtained. These matrix polynomials of dimension $N \times N$ contain all of the information produced by the coherence phenomena in the multimode-multilayer system. Transmission and conductance probabilities have been calculated for one- and two-channel problems. In the one-channel case, the total transmission $|t_n|^2$ and the four-probe Landauer conductance $G_n$ were plotted as functions of the Bargmann parameters, leaving the example as arbitrary as possible. To illustrate the use of our method with more than one open channel, we considered a 3D confined semiconductor with equidistant planes of $\delta$-potential scatterer centres, and we have calculated transmission probabilities $T_{n,ij} = |t_{n,ij}|^2$ from channel $i$ to channel $j$, for coupled and uncoupled channels. Strong interference effects, like resonance suppression and new resonant levels, appear because of channel coupling. Concerning the non-commutative polynomials, we emphasized their relation with the resonant character of the transport physical quantities and the multiple interference phenomena occurring in these locally periodic systems.

## Acknowledgments

The author gratefully acknowledges Rolando Pérez-Alvarez, Alfonso Anzaldo-Meneses and Arturo Robledo-Martínez for useful and clarifying comments. This work has been supported partially by the ICTP, Trieste, and CONACyT of México under project No 350-E9301.

## References

[1] Tsu R and Esaki L 1993 *Appl. Phys. Lett.* **24** 562
Mendez E E *et al* 1985 *Appl. Phys. Lett.* **47** 415
Mendez E E *et al* 1991 *Phys. Rev.* B. **43** 5196
[2] Ivchenko E L and Pikus G E 1997 *Superlattices and other Heterostructures. Symmetry and Optical Phenomena* 2 edn, trans. G P Skrebtsov (New York: Springer)
[3] Helm M Jr *et al* 1989 *Phys. Rev. Lett.* **63** 74
Grahn H T *et al* 1990 *Phys. Rev. Lett.* **64** 3163
Morifuji M and Hamaguchi C 1995 *Phys. Rev.* B **52** 14 131
Chen P A and Chang C Y 1993 *J. Appl. Phys.* **74** 7294
Capasso F *et al* 1990 *Physics of Quantum Electron Devices* ed F Capasso (Berlin: Springer) pp 181–2
[4] Vezzetti D J and Cahay M 1986 *J. Phys. D: Appl. Phys.* **19** L53–L55
Griffiths D J and Taussing N F 1992 *Am. J. Phys.* **60** 883
[5] See, for example, Beenakker C W J and van Houten H 1991 *Solid State Phys.* **44** 1
Ulloa S E, MacKinnon A, Castaño E and Kirczenow G 1992 *Handbook on Semiconductors* ed P T Landsberg (Amsterdam: North-Holland)
[6] Mello P A, Pereyra P and Kumar N 1988 *Ann. Phys.* **181** 290
[7] Bagwell P F *et al* 1990 *Phys. Rev.* B. **41** 10 354
[8] Anzaldo-Meneses A and Pereyra P, to be published

[9] Pereyra P 1995 *J. Math. Phys.* **36** 1166
[10] Gelfand I M *et al* 1995 *Adv. Math.* **112** 218
[11] Kolatas T H and Lee A R 1991 *Eur. J. Phys.* **12** 275
Lee H W, Zysnarsky A and Kerr P 1989 *Am. J. Phys.* **57** 729
Pérez-Alvarez R and Rodriguez-Coppola H 1988 *Phys. Status Solidi* b **145** 493
Sprung D W, Wu H and Martorell J 1993 *Am. J. Phys.* **61** 1118
Rozman M G, Reineker P and Tehver R 1994 *Phys. Lett.* **187A** 127